

Spectral analysis of a selected non self-adjoint Hamiltonian in an infinite dimensional Hilbert space

N. Bebiano*, J. da Providência[†] and J.P. da Providência[‡]

September 3, 2015

Abstract

The so-called equation of motion method is useful to obtain the explicit form of the eigenvectors and eigenvalues of certain non self-adjoint bosonic Hamiltonians with real eigenvalues. These operators can be diagonalized when they are expressed in terms of pseudo-bosons, which do not behave as ordinary bosons under the adjoint transformation, but obey the Weil-Heisenberg commutation relations.

1 Introduction and preliminaries

In conventional formulations of non-relativistic quantum mechanics, the Hamiltonian operator is self-adjoint. However, certain relativistic extensions of quantum mechanics lead to the consideration of non self-adjoint Hamiltonian operators with a real spectrum. This motivated an intense research activity, both on the physical and mathematical level (see, e.g. [2, 4, 5, 6, 7, 8, 9] and their references).

Throughout, we shall use synonymously the terms Hermitian and self-adjoint. Denote by $L^2(\mathbb{R}^2)$ the Hilbert space of square integrable functions in two real variables, endowed with the standard inner product

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \overline{f(x, y)} dx dy.$$

*CMUC, University of Coimbra, Department of Mathematics, P 3001-454 Coimbra, Portugal (bebiano@mat.uc.pt)

[†]CFisUC, University of Coimbra, Department of Physics, P 3004-516 Coimbra, Portugal (providencia@teor.fis.uc.pt)

[‡]Departamento de Física, Univ. of Beira Interior, P-6201-001 Covilhã, Portugal (joaodaprovidencia@daad-alumni.de)

For

$$\mathcal{D} = \left\{ f(x, y) e^{-(x^2+y^2)} : f(x, y) \text{ is a polynomial in } x, y \right\} \quad (1)$$

which is a dense domain in $L^2(\mathbb{R}^2)$ [3], let $a, b : \mathcal{D} \rightarrow \mathcal{D}$ be bosonic operators defined, respectively, by

$$a = x + \frac{1}{2} \frac{\partial}{\partial x}, \quad b = y + \frac{1}{2} \frac{\partial}{\partial y}.$$

Consider also their adjoints $a^*, b^* : \mathcal{D} \rightarrow \mathcal{D}$

$$a^* = x - \frac{1}{2} \frac{\partial}{\partial x}, \quad b^* = y - \frac{1}{2} \frac{\partial}{\partial y}.$$

We recall that, conventionally, a, b are said to be *annihilation operators*, while a^*, b^* are *creation operators*. It is worth noticing that these operators are unbounded. Moreover, \mathcal{D} is stable under the action of a, b and of their adjoints, and they satisfy the commutation rules (CR's),

$$[a, a^*] = [b, b^*] = \mathbf{1}, \quad (2)$$

where $\mathbf{1}$ is the identity operator in \mathcal{D} . (This means that $aa^*f - a^*af = bb^*f - b^*bf = f$ for any $f \in \mathcal{D}$.) Furthermore,

$$[a, b^*] = [b, a^*] = [a^*, b^*] = [a, b] = 0. \quad (3)$$

As it is well-known, the canonical commutation relations (2) and (3) characterize an algebra of Weil-Heisenberg (W-H). Moreover, the following holds,

$$a\Phi_0 = b\Phi_0 = 0,$$

for $\Phi_0 = e^{-(x^2+y^2)} \in \mathcal{D}$, a so-called *vacuum state*. The set of vectors

$$\{\Phi_{m,n} = a^{*m} b^{*n} \Phi_0 : m, n \geq 0\}, \quad (4)$$

constitutes a basis of \mathcal{H} , that is, every vector in $L^2(\mathbb{R}^2)$ can be uniquely expressed in terms of this vector system, which is *complete*, since 0 is the only vector orthogonal to all its elements.

The main goal of this note is to investigate spectral properties of certain non self-adjoint operators which are expressed as quadratic combinations of bosonic operators.

2 Non self-adjoint Hamiltonian constructed in terms of $su(1, 1)$ generators

A linear operator acting on a finite dimensional Hilbert space which is non self-adjoint and has distinct real eigenvalues, is similar to its adjoint operator. Concretely, the following holds, having in mind that the spectrum of a finite matrix is the set of its eigenvalues.

Theorem 2.1 *Let H be an $n \times n$ complex non self-adjoint matrix with distinct real eigenvalues. Then H and H^* have a common spectrum, a complete systems of eigenvectors and they are similar, that is, there exists a unitary matrix S such that $H^* = SHS^{-1}$. Moreover, if $H\phi_i = \lambda_i\phi_i$, $H^*\psi_i = \lambda_i\psi_i$, $i = 1, \dots, n$, then the eigenvectors may be normalized so that $\langle \psi_i, \phi_j \rangle = \delta_{ij}$.*

Proof. Since the eigenvalues of H are real, they coincide with those of H^* . Let $\sigma(H) = \sigma(H^*) = \{\lambda_1, \dots, \lambda_n\}$, and let ϕ_i, ψ_i , $i = 1, \dots, n$ be such that

$$H\phi_i = \lambda_i\phi_i, \quad H^*\psi_i = \lambda_i\psi_i.$$

Further, as the eigenvalues are mutually distinct, the corresponding eigenvectors are linearly independent, and so the eigensystems $\{\psi_i\}$, $\{\phi_j\}$ corresponding to the eigenvalues λ_i , $i = 1, 2, \dots, n$, are complete.

Let S be defined by $\psi_i = S\phi_i$. Thus, S is unitary and

$$H^*S\phi_i = \lambda_i S\phi_i = SH\phi_i, \quad i = 1, \dots, n,$$

implying that H and H^* are unitarily similar.

Having in mind that

$$\langle H\phi_i, \psi_j \rangle = \langle \phi_i, H^*\psi_j \rangle = \lambda_i \langle \phi_i, \psi_j \rangle = \lambda_j \langle \phi_i, \psi_j \rangle,$$

the orthonormality relation follows. ■

The following question naturally arises. Do the above properties survive in the infinite dimensional setting? We answer this question for the model described by the linear operator $H : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ defined as

$$H = a^*a + bb^* + \beta(a^*a - bb^*) + \gamma(a^*b^* - ab), \quad \beta, \gamma \in \mathbb{R}, \quad (5)$$

which treats a system of two interacting bosons. It is obvious that H is non self-adjoint.

Consider the class \mathcal{C} of unbound linear operators H acting on the infinite dimensional Hilbert space \mathcal{H} and satisfying the following properties:

- (I) H is non self-adjoint,
- (II) H has real eigenvalues,
- (III) H and H^* are isospectral.
- (IV) The associated eigenvectors form biorthogonal bases of \mathcal{H} .
- (V) There exists a unitary transformation S on \mathcal{H} such that $H^* = SHS^{-1}$.

Throughout we prove that H in (5) belongs to the class \mathcal{C} , and so, in the eigenvectors bases, H and H^* have a diagonal representation.

3 The equation of motion method for pseudo-bosonic operators

Let us consider a non self-adjoint Hamiltonian H ($H \neq H^*$) which is expressed in terms of unbounded bosonic operators $a_1, \dots, a_n, a_1^*, \dots, a_n^*$, that is, linear operators acting on \mathcal{D} and satisfying the CR's,

$$[a_i, a_j^*] = \delta_{ij}, [a_i, a_j] = 0, [a_i^*, a_j^*] = 0. \quad (6)$$

As usual, δ_{ij} denotes the Kroenecker symbol (which equals 0 for $i \neq j$ and 1 for $i = j$).

Let \mathcal{V} be the linear space spanned by $a_1, \dots, a_n, a_1^*, \dots, a_n^*$ and let $f \in \mathcal{V}$, that is, f is a linear combination of the operators $a_1, \dots, a_n, a_1^*, \dots, a_n^*$. Having in mind that

$$[a_i a_j, a_k^*] = \delta_{ik} a_j + \delta_{jk} a_i, [a_i^* a_j, a_k] = \delta_{ik} a_j$$

and their adjoint relations, it is clear that $[H, f] \in \mathcal{V}$. Obviously, the operator $[H, f] : \mathcal{V} \rightarrow \mathcal{V}$ is a linear operator on f and next we prove that its eigenvalues are closely related with those of H .

Theorem 3.1 *Let H be a non self-adjoint operator acting on \mathcal{H} with real eigenvalues expressed in terms of boson operators satisfying (6). Let $f \in \mathcal{V}$ and $\lambda \in \mathbb{R}$ be such that*

$$[H, f] = \lambda f. \quad (7)$$

Suppose $\Lambda \in \mathbb{R}$ and $\psi \in \mathcal{H}$ are, respectively, an eigenvalue of H and an associated eigenvector. If $f\psi \neq 0$, then also $\Lambda + \lambda$ is an eigenvalue of H and $f\psi$ a corresponding eigenvector. Moreover, there exists $0 \neq g \in \mathcal{V}$ such that

$$[H, g] = -\lambda g.$$

Proof. Under the hypothesis, we have $H\psi = \Lambda\psi$. Then,

$$Hf\psi = [H, f]\psi + fH\psi = (\Lambda + \lambda)f\psi.$$

Since Λ and $\Lambda + \lambda$ are both real, so is λ . Since $[H^*, f^*] = -\lambda f^*$ and H and H^* have the same eigenvalues, it follows that there exists $0 \neq g \in \mathcal{V}$ such that $[H, g] = -\lambda g$. Observe that the operator $[H, f] : \mathcal{V} \rightarrow \mathcal{V}$ is represented by some $2n \times 2n$ matrix T , while the operator $[H^*, f^*] : \mathcal{V} \rightarrow \mathcal{V}$ is represented by the transconjugate matrix T^* , and so the eigenvalues of T and T^* coincide since they are real. ■

If $\lambda > 0$, f is called an *excitation operator* (*deexcitation operator* if $\lambda < 0$).

The so called equation of motion method (EMM) consists in the solution of eq. (7), with the determination of the eigenvalues of H and corresponding eigenvectors.

Let $0 \neq g \in \mathcal{V}$ be such that $[H, g] = \lambda'g$. The Jacobi identity yields

$$[[H, f], g] + [[f, g], H] + [[g, H], f] = 0.$$

Now, $[[f, g], H] = 0$, since $[f, g]$ is a multiple of the identity. Thus,

$$(\lambda + \lambda')[f, g] = 0.$$

As a consequence, either $\lambda = -\lambda'$ or $[f, g] = 0$. If $[f, g] \neq 0$, which happens only if $\lambda = -\lambda'$, f and g may be normalized so that $[f, g] = 1$. Since $f^* \neq g$, the operators obtained by this procedure do not describe “ordinary” bosons. In this case, it may be seen that there exists $0 \neq \Psi_0 \in \mathcal{D}$ such that either $f\Psi_0 = 0$ or $g\Psi_0 = 0$. Assume that $f\Psi_0 = 0$. We write $g = f^\dagger$ and it can be checked that the vectors system $\{f^{\dagger n}\Psi_0 : n \geq 0\}$ constitutes a complete basis of $L^2(\mathbb{R}^2)$. There also exists $\Psi'_0 \in \mathcal{D}$ such that $g^*\Psi'_0 = f^{\dagger*}\Psi'_0 = 0$ and the vector system $\{f^{*n}\Psi'_0 : n \geq 0\}$ is an eigenvector system for H^* , which constitutes a complete basis of $L^2(\mathbb{R}^2)$. The bases

$$\{f^{\dagger n}\Psi_0 : n \geq 0\} \quad \text{and} \quad \{f^{*n}\Psi'_0 : n \geq 0\}$$

are biorthogonal:

$$\langle f^{*n}\Psi'_0, f^{\dagger m}\Psi_0 \rangle = \langle \Psi'_0, f^n f^{\dagger m}\Psi_0 \rangle = \delta_{nm} m! \langle \Psi'_0, \Psi_0 \rangle.$$

Thus, the operators f, f^\dagger are said to describe *pseudo-bosons* for H , while the operators $f^{\dagger*}, f^*$ describe pseudo-bosons for H^* (cf. Bagarello [1, 2]). If $g\Psi_0 = 0$, with $\Psi_0 \in \mathcal{D}$, an analogous discussion takes place.

4 H belongs to the class \mathcal{C}

Throughout we prove that H in eq. (5) belongs to the class \mathcal{C} . As already observed, H satisfies (I).

(II) H has real eigenvalues.

The operator H is quadratic in the bosonic operators and we may explicitly find its eigenvalues by the equations of motion method, as follows. We look for a linear combination of the bosonic operators such that its commutator with H satisfies the proportionality condition,

$$[H, (x_a a^* + x_b b^* + y_a a + y_b b)] = \lambda(x_a a^* + x_b b^* + y_a a + y_b b), \quad \lambda \in \mathbb{R}.$$

This leads to the following system of linear equations in x_a, x_b, y_a, y_b and λ :

$$\begin{aligned} (1 + \beta)x_a - \gamma y_b &= \lambda x_a, \\ (1 - \beta)x_b - \gamma y_a &= \lambda x_b, \\ -(1 + \beta)y_a - \gamma x_b &= \lambda y_a, \\ -(1 - \beta)y_b - \gamma x_a &= \lambda y_b. \end{aligned}$$

Solving this linear system, we readily obtain $\lambda = \pm\beta \pm \sqrt{1+\gamma^2}$. The vectors $(x_a, x_b, y_a, y_b)^T$ and the parameters λ are, respectively, the eigenvectors and eigenvalues of the 4×4 real symmetric matrix

$$T = \begin{bmatrix} 1+\beta & 0 & 0 & -\gamma \\ 0 & 1-\beta & -\gamma & 0 \\ 0 & -\gamma & -1-\beta & 0 \\ -\gamma & 0 & 0 & -1+\beta \end{bmatrix}.$$

The eigenvalues of T and associated eigenvectors are given by

$$\begin{aligned} \lambda_1 &= -\beta - \sqrt{1+\gamma^2}, & v_1 &= \mathcal{N}(0, -1 + \sqrt{1+\gamma^2}, \gamma, 0)^T, \\ \lambda_2 &= \beta - \sqrt{1+\gamma^2}, & v_2 &= \mathcal{N}(-1 + \sqrt{1+\gamma^2}, 0, 0, \gamma)^T, \\ \lambda_3 &= -\beta + \sqrt{1+\gamma^2}, & v_3 &= \mathcal{N}(0, 1 + \sqrt{1+\gamma^2}, -\gamma, 0)^T, \\ \lambda_4 &= \beta + \sqrt{1+\gamma^2}, & v_4 &= \mathcal{N}(1 + \sqrt{1+\gamma^2}, 0, 0, -\gamma)^T. \end{aligned}$$

The determination of the normalization constant \mathcal{N} is postponed and done according to future convenience.

Consider now the operators

$$\begin{aligned} c &= \mathcal{N}((-1 + \sqrt{1+\gamma^2})b^* + \gamma a), \\ d &= \mathcal{N}((-1 + \sqrt{1+\gamma^2})a^* + \gamma b), \\ d^\dagger &= \mathcal{N}((1 + \sqrt{1+\gamma^2})b^* - \gamma a), \\ c^\dagger &= \mathcal{N}((1 + \sqrt{1+\gamma^2})a^* - \gamma b), \end{aligned}$$

also defined on \mathcal{D} . Notice that $c^\dagger \neq c^*$ and $d^\dagger \neq d^*$.

Let us take

$$\mathcal{N} = \left(2\gamma\sqrt{1+\gamma^2}\right)^{-1/2}.$$

Then, the operators $c^\dagger, d^\dagger, c, d$, satisfy the CR's of a Weil-Heisenberg algebra,

$$[c, c^\dagger] = [d, d^\dagger] = \mathbf{1},$$

vanishing all the remaining commutators between them. Moreover, H can be written as

$$H = \beta(c^\dagger c - d^\dagger d) + \sqrt{1+\gamma^2}(c^\dagger c + dd^\dagger).$$

Next, we show that the Hamiltonian presents a diagonal form when it is expressed in terms of the operators $c^\dagger, d^\dagger, c, d$.

Having in mind that

$$\Phi_0 = \exp(-(x^2 + y^2)) \in \mathcal{D}$$

satisfies $a\Phi_0 = b\Phi_0 = 0$, it may be easily verified that

$$\Psi_0 = \exp(-\alpha a^* b^*) \Phi_0 = \exp\left(-\frac{1+\alpha^2}{1-\alpha^2}(x^2 + y^2) - \frac{4\alpha}{1-\alpha^2}xy\right),$$

where $\alpha = \gamma/(1 + \sqrt{1 + \gamma^2}) = (-1 + \sqrt{1 + \gamma^2})/\gamma$, satisfies

$$c\Psi_0 = d\Psi_0 = 0.$$

Of course, $\Psi_0 \in \text{span}\mathcal{D}$. Let

$$\{\Psi_{m,n} = c^{\dagger m} d^{\dagger n} \Psi_0 : m, n \geq 0\}.$$

From the CR's of a Weil-Heisenberg algebra, it follows that

$$H\Psi_{m,n} = \left(\sqrt{1+\gamma^2} + m(\beta + \sqrt{1+\gamma^2}) + n(-\beta + \sqrt{1+\gamma^2})\right) \Psi_{m,n}.$$

Henceforth, the eigenvalues of H are given by

$$E_{m,n} = \sqrt{1+\gamma^2} + m(\beta + \sqrt{1+\gamma^2}) + n(-\beta + \sqrt{1+\gamma^2}). \quad (8)$$

Thus, (II) holds. Moreover, the associated eigenvectors to E_{mn} are

$$\Psi_{m,n} = c^{\dagger m} d^{\dagger n} \Psi_0, \quad m, n \geq 0.$$

These eigenvectors constitute a basis of \mathcal{H} . Further, it can be shown that this basis is complete [3].

In order to prove that H^* and H have the same eigenvalues, let us consider the state vector

$$\Psi'_0 = \exp(\alpha a^* b^*) \Phi_0.$$

Clearly,

$$c^{\dagger*} \Psi'_0 = d^{\dagger*} \Psi'_0 = 0.$$

It follows that the state vectors

$$\Psi'_{m,n} = c^{*m} d^{*n} \Psi'_0, \quad m, n \geq 0,$$

are eigenvectors of H^* ,

$$H^* \Psi'_{m,n} = \left(\sqrt{1+\gamma^2} + m(\beta + \sqrt{1+\gamma^2}) + n(-\beta + \sqrt{1+\gamma^2})\right) \Psi'_{m,n}.$$

Hence, H^* and H are isospectral and (III) holds.

Next, we show that the basis constituted by the eigenvectors of H is orthogonal to the basis of the eigenvectors of H^* . The basis

$$\{\Psi'_{m,n} = c^{*m} d^{*n} \Psi'_0 : m, n \geq 0\},$$

is orthogonal to the basis $\{\Psi_{m,n} : m, n \geq 0\}$, as we have

$$\langle \Psi_{m,n}, \Psi'_{p,q} \rangle = \langle c^{\dagger m} d^{\dagger n} \Psi_0, c^{*p} d^{*q} \Psi'_0 \rangle = \langle c^p d^q c^{\dagger m} d^{\dagger n} \Psi_0, \Psi'_0 \rangle = m!n! \delta_{mp} \delta_{n,q} \langle \Psi_0, \Psi'_0 \rangle.$$

Henceforth, (IV) is satisfied.

Finally, (V) holds. The unitary transformation

$$S = \exp \left(-i \frac{\pi}{2} (a^* a + b^* b) \right),$$

acts on a and b as follows:

$$a \rightarrow ia = SaS^{-1}, \quad b \rightarrow ib = SbS^{-1},$$

and so S takes H into H^* , that is,

$$H^* = SHS^{-1}.$$

Hence, H satisfies (V).

Some considerations are in order.

1. In the above considered eigenvectors system, the matrix of H has the following representation

$$\begin{aligned} & \begin{bmatrix} \rho & 0 & 0 & 0 & \dots \\ 0 & \beta + 2\rho & 0 & 0 & \dots \\ 0 & 0 & 2\beta + 3\rho & 0 & \dots \\ 0 & 0 & 0 & 3\beta + 4\rho & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \otimes I \\ & + I \otimes \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & -\beta + \rho & 0 & 0 & \dots \\ 0 & 0 & 2(-\beta + \rho) & 0 & \dots \\ 0 & 0 & 0 & 3(-\beta + \rho) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \end{aligned}$$

where I is the semi-infinite identity matrix, $\rho = \sqrt{1 + \gamma^2}$ and \otimes denotes the usual Kroenecker product. Hence, the matrix of H is the following diagonal matrix

$$\bigoplus_{k=0}^{\infty} \begin{bmatrix} k(\beta + \rho) + \rho & 0 & 0 & \dots \\ 0 & k(\beta + \rho) - \beta + 2\rho & 0 & \dots \\ 0 & 0 & k(\beta + \rho) - 2\beta + 3\rho & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

2. In the consstruced eigenvectors bases, H and H^* have a diagonal representation.

3. The integers m, n are the number of pseudo-bosons of type c^\dagger, d^\dagger , respectively. Since, however, $c^\dagger \neq c^*$, $d^\dagger \neq d^*$, they do not describe “ordinary” dynamical bosons and they are said to describe dynamical pseudo-bosons [1, 2].

Remark that, although H is non self-adjoint, it has real eigenvalues and a complete system of eigenvectors for any $\gamma, \beta \in \mathbb{R}$. If we replace $\gamma \in \mathbb{R}$ by $i\lambda$, H becomes self-adjoint, but then it only has real eigenvalues and a complete system of eigenvectors if $|\gamma| < 1$. The existence of complex eigenvalues for $|\gamma| > 1$ indicates that the system under consideration is not stable, or that H is not bounded from below, violating conditions which, in general, are required on physical grounds.

5 Invariant subspaces

In this Section, we provide an alternative approach to determine the eigenvalues and eigenvectors of H .

We notice that the operator H is expressed as a linear combination of the generators of the $su(1, 1)$ algebra

$$a^*a + bb^*, \quad ab, \quad a^*b^*,$$

and of the operator

$$a^*a - b^*b,$$

called, for simplicity, *Casimir operator*, since it is closely related to the actual Casimir operator which is given by

$$C := (a^*a + bb^*)^2 - 2a^*b^*ab - 2aba^*b^* = (a^*a - b^*b - \mathbf{1})(a^*a - b^*b + \mathbf{1}).$$

This is the key fact for the alternative procedure for the spectral analysis in this Section.

Since any $f \in L^2(\mathbb{R}^2)$ may be expanded in the basis $\{\Phi_{mn} : m, n \geq 0\}$, we may identify $L^2(\mathbb{R}^2)$ with $\mathcal{H} = \text{span}\{\Phi_{mn} : m, n \geq 0\}$. It is convenient to express \mathcal{H} as a direct sum of eigenspaces of the Casimir operator,

$$\mathcal{H} = \bigoplus_{k=-\infty}^{+\infty} \mathcal{H}_k, \quad \mathcal{H}_k = \text{span}\{\Phi_{mn}; m - n = k, m, n \geq 0\}.$$

Notice that eigenspaces of the Casimir operator are invariant subspaces of the generators of the $su(1, 1)$ algebra, and consequently of H . In \mathcal{H}_k , the generator a^*b^* is matricially represented by

$$A_+ = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{|k|+1} & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2(|k|+2)} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3(|k|+3)} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

while the generator ab is represented by

$$A_- = A_+^T,$$

and the generator $a^*a + bb^*$ is represented by

$$A_0 = \begin{bmatrix} |k|+1 & 0 & 0 & 0 & \dots \\ 0 & |k|+3 & 0 & 0 & \dots \\ 0 & 0 & |k|+5 & 0 & \dots \\ 0 & 0 & 0 & |k|+7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following CR's of the $su(1,1)$ algebra are satisfied

$$[A_-, A_+] = A_0, \quad [A_0, A_+] = 2A_+, \quad [A_0, A_-] = -2A_-. \quad (9)$$

The operator A_+ (A_-) is said to be a *raising* (*lowering*) operator. That is, if Φ is an eigenvector of A_0 , so that $A_0\Phi = \Lambda\Phi$, then $A_+\Phi$ is an eigenvector associated with an upward shifted eigenvalue,

$$A_0A_+\Phi = (\Lambda + 2)A_+\Phi.$$

Similarly, $A_-\Phi \neq 0$ is an eigenvector associated with a downward shifted eigenvalue,

$$A_0A_-\Phi = (\Lambda - 2)A_-\Phi.$$

The spectrum of A_0 is bounded from below and the eigenvector $\Phi_0 = (1, 0, 0, \dots)^T$ satisfies $A_-\Phi_0 = 0$, being called a *lowest weight state*. A set of eigenvectors associated with eigenvalues of A_0 , which are positive, is obtained by acting successively with A_+ on Φ_0 . We observe that $A_0 = A_0^*$ and $A_+ = A_-^*$. It is interesting that, in \mathcal{H}_k , the Hamiltonian H is represented by a so called *pseudo-Jacobi matrix*, that is, a Jacobi matrix pre or pos multiplied by the involution matrix $\text{diag}(1, -1, 1, -1, \dots)$,

$$H = \begin{bmatrix} \beta k + |k| + 1 & -\gamma\sqrt{|k|+1} & 0 & 0 & \dots \\ \gamma\sqrt{|k|+1} & \beta k + |k| + 3 & -\gamma\sqrt{2(|k|+2)} & 0 & \dots \\ 0 & \gamma\sqrt{2(|k|+2)} & \beta k + |k| + 5 & -\gamma\sqrt{3(|k|+3)} & \dots \\ 0 & 0 & \gamma\sqrt{3(|k|+3)} & \beta k + |k| + 7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

For simplicity, this matrix has been denoted by the same symbol H . It is clear that H is unitarily similar to its transpose H^T ,

$$H^T = e^{i(\pi/2)A_0} H e^{-i(\pi/2)A_0}.$$

The diagonalization of this matrix provides an alternative procedure to the previous approach.

For completeness, it may be in order to observe that it is also possible to represent the generator ab by

$$A'_+ = \begin{bmatrix} 0 & \sqrt{|k|+1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2(|k|+2)} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3(|k|+3)} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

the generator a^*b^* by

$$A'_- = A'_+{}^T,$$

and the generator $-a^*a - bb^*$ by

$$A'_0 = \begin{bmatrix} -|k|-1 & 0 & 0 & 0 & \dots \\ 0 & -|k|-3 & 0 & 0 & \dots \\ 0 & 0 & -|k|-5 & 0 & \dots \\ 0 & 0 & 0 & -|k|-7 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The commutation relations $[A'_-, A'_+] = A_0$, $[A'_0, A'_+] = 2A'_+$, $[A'_0, A'_-] = -2A'_-$, which characterize the $su(1, 1)$ algebra, are clearly satisfied. However, in this case, the role of the lowest weight state is played by a *highest weight state*, such that $A_+\Phi_0 = 0$.

Let \mathcal{V} be the linear space spanned by A_0, A_+, A_- and let

$$H_0 = H - \beta k I = A_0 + \gamma(A_+ - A_-).$$

Obviously, $H_0 \in \mathcal{V}$. Let

$$[H_0, \mathcal{V}] := \{H_0x - xH_0 : x \in \mathcal{V}\}.$$

Consider the linear operator $[H_0, \mathcal{V}] : \mathcal{V} \rightarrow \mathcal{V}$. We look for $xA_+ + yA_- + zA_0$ such that

$$[H_0, (xA_+ + yA_- + zA_0)] = \lambda(xA_+ + yA_- + zA_0).$$

The eigenvector $xA_+ + yA_- + zA_0$ is easily determined by the secular equation

$$\begin{bmatrix} 2 & 0 & -2\gamma \\ 0 & -2 & -2\gamma \\ -\gamma & -\gamma & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

The eigenvalues and respective eigenvectors of this 3×3 matrix are given by

$$2\sqrt{1+\gamma^2}, \quad \left(\frac{1+\sqrt{1+\gamma^2}}{\gamma}, \frac{\gamma}{1+\sqrt{1+\gamma^2}}, -1 \right)^T,$$

$$-2\sqrt{1+\gamma^2}, \quad \left(\frac{-1+\sqrt{1+\gamma^2}}{\gamma}, \frac{\gamma}{-1+\sqrt{1+\gamma^2}}, 1 \right)^T,$$

and, of course, $0, \quad (\gamma, -\gamma, 1)^T$. Let us consider now the matrices

$$\begin{aligned} B_+ &= \frac{\gamma}{2\sqrt{1+\gamma^2}} \left(\frac{1+\sqrt{1+\gamma^2}}{\gamma} A_+ + \frac{\gamma}{1+\sqrt{1+\gamma^2}} A_- - A_0 \right) \\ B_- &= \frac{\gamma}{2\sqrt{1+\gamma^2}} \left(\frac{-1+\sqrt{1+\gamma^2}}{\gamma} A_+ + \frac{\gamma}{-1+\sqrt{1+\gamma^2}} A_- + A_0 \right), \\ B_0 &= \frac{1}{\sqrt{1+\gamma^2}} (A_0 + \gamma(A_+ - A_-)). \end{aligned}$$

The following CR's of the $su(1, 1)$ algebra are satisfied

$$[B_-, B_+] = B_0, \quad [B_0, B_+] = 2B_+, \quad [B_0, B_-] = -2B_-,$$

suggesting that

$$\sigma(H_0) = \sqrt{1+\gamma^2} \{|k|+1, |k|+3, |k|+5, \dots\}, \quad (10)$$

in agreement with the previous results in Section 4. We remark that $B_0 \neq B_0^*$, $B_+ \neq B_-^*$. The operator B_+ (B_-) is said to be a raising (lowering) operator. That is, if Ψ is an eigenvector of B_0 , so that $B_0\Psi = \Lambda\Psi$, then $B_+\Psi$ is an eigenvector associated with an upward shifted eigenvalue, $B_0B_+\Psi = (\Lambda+2)B_+\Psi$. Similarly, $B_-\Psi \neq 0$ is an eigenvector associated with a downward shifted eigenvalue, $B_0B_-\Psi = (\Lambda-2)B_-\Psi$. Next, we observe that

$$\Psi_0 = \left(1, -\alpha(|k|+1)^{1/2}, \alpha^2 \binom{|k|+2}{2}^{1/2}, -\alpha^3 \binom{|k|+3}{3}^{1/2}, \dots \right)^T$$

satisfies $B_-\Psi_0 = 0$, as may be easily checked. It follows that the spectrum of B_0 is bounded from below. A laborious but straightforward computation shows that the Casimir operator reduces to

$$C := B_0^2 - 2(B_+B_- + B_-B_+) = B_0^2 - 2B_0 - 4B_+B_- = (k^2 - 1)I.$$

Thus,

$$C\Psi_0 = (B_0^2 - 2B_0)\Psi_0 = (k^2 - 1)\Psi_0,$$

implying that

$$B_0\Psi_0 = (|k|+1)\Psi_0,$$

so that

$$\sigma(B_0) = \{|k|+1, |k|+3, |k|+5, \dots\}.$$

This confirms (10).

6 Discussion

In Section 2, the diagonalization of non-Hermitian Hamiltonians with real eigenvalues, which are expressed as quadratic combinations of bosonic operators, is briefly analyzed. It is shown that, quite generally, Hamiltonians of this class are diagonalizable in terms of dynamical pseudo-bosons, which are determined by the EMM. In Section 3, a non self-adjoint Hamiltonian which is expressed as a linear combination of $su(1,1)$ generators, is investigated. Its eigenvalues and eigenvectors have been determined with the help of a matrix T of size 4 that is real Hermitian and is determined by the EMM. The investigated Hamiltonian has a complete system of eigenvectors expressed in terms of the creation and annihilation operators of pseudo-bosons, and is orthogonal to the system of eigenvectors of the adjoint Hamiltonian. Complete vector systems constructed in terms of boson creation operators acting on the associated vacuum state are obtained. Infinite matrix representations of the Hamiltonian in such systems are presented.

It would be interesting to consider more general Hamiltonians of the investigated type. A challenging problem would be to analyze the existence of infinite dimensional versions in the spirit of Theorem 2.1

References

- [1] F. Bagarello, Construction of pseudo-bosons systems, J. Math. Phys. **51** (2010) 023531; Pseudo-bosons, so far, Reports on Mathematical Physics, 68 (2011) 175-210; More mathematics for pseudo-bosons, J. Math. Phys., 54 (2013) 063512.
- [2] F. Bagarello, F. Gargano, D. Volpe, \mathcal{D} -deformed harmonic oscillators, International Journal of Theoretical Physics 01/2015.
- [3] N. Bebiano, J. da Providência and JP. da Providência, Hamiltonians expressed in terms of bosonic operators and their spectra, submitted.
- [4] J. da Providência, N. Bebiano and JP. da Providência, Non Hermitian operators with real spectra in Quantum Mechanics, Brazilian Journal of Physics, 41 (2011) 78-85.
- [5] F.G. Scholtz, H.B. Geyer and F.J.W. Hahne, Quasi-Hermitian operators in quantum mechanics and the variational principle, Ann. Phys. NY 213 (1992) 74.
- [6] C.M. Bender and S. Boettcher, Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry , Phys. Rev. Lett., 80 (1998) 5243-5246, C.M. Bender,

- D.C. Brody and H.F. Jones, Complex Extension of Quantum Mechanics, Phys. Rev. Lett, 89 (2002) 27041.
- [7] A. González López and T. Tanaka, Nonlinear pseudo-supersymmetry in the framework of N -fold supersymmetry, J. Phys. A: Math. Gen. 39 (2006) 3715-23; T. Tanaka, Preprint quant-ph/0603075; T. Tanaka, J. Phys. A: Math. Gen. 39 (2006) L369-L376.
- [8] M. Znojil, Should PT Symmetric Quantum Mechanics Be Interpreted as Nonlinear?, J. Nonlin. Math. Phys., 9 (2002) 122-123.
- [9] A Mostafazadeh, Exact PT-symmetry is equivalent to Hermiticity, J. Phys. A: Math. Gen. 36 (2003) 7081; Complex Extension of Quantum Mechanics, J. Math. Phys. 46 (2005) 102108; Delta-function potential with a complex coupling, J. Phys. A: Math. Gen. 39 (2006) 13495; Pseudo-Hermiticity and generalized PT- and CPT- symmetries, J. Math. Phys. 44 (2003) 974; arXiv:quant-ph/060173v2.